

ABSTRACT MATRIX-TREE THEOREM AND BERNARDI POLYNOMIAL

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ABSTRACT. This paper is a continuation of [2]. We prove a three-parameter family of identities (Theorem 1.1) involving a version of the Tutte polynomial for directed graphs introduced by Awan and Bernardi [1]. A particular case of this family (Corollary 1.6) is the higher-degree generalization of the matrix-tree theorem proved in [2], which thus receives a new proof, shorter (and less direct) than the original one. The theory has a parallel version for undirected graphs (Theorem 1.2).

1. DEFINITIONS AND MAIN RESULTS

The following theory has two parallel versions — for directed and undirected graphs — so let us introduce notation for both cases.

By $\Gamma_{n,k}$ we denote the set of directed graphs with n vertices numbered $1, \dots, n$ and k edges numbered $1, \dots, k$; in other words, an element $G \in \Gamma_{n,k}$ is a k -element sequence $([a_1, b_1], \dots, [a_k, b_k])$ where $a_1, \dots, a_k, b_1, \dots, b_k \in \{1, \dots, n\}$. Loops (edges $[a, a]$) and parallel edges (pairs $[a_i, b_i] = [a_j, b_j]$) are allowed. Similarly, by $\Upsilon_{n,k}$ we denote the set of unoriented graphs with n numbered vertices and k numbered edges: $\Upsilon_{n,k} = \{(\{a_1, b_1\}, \dots, \{a_k, b_k\})\}$. The forgetful map $|\cdot| : \Gamma_{n,k} \rightarrow \Upsilon_{n,k}$ relates to every graph G its undirected version $|G|$ obtained by dropping the edge orientation: $[a, b] \mapsto \{a, b\}$.

Denote by $\mathcal{G}_{n,k}$ (resp., $\mathcal{Y}_{n,k}$) a vector space over \mathbb{C} spanned by $\Gamma_{n,k}$ (resp., $\Upsilon_{n,k}$). The forgetful map is naturally extended to the linear map $|\cdot| : \mathcal{G}_{n,k} \rightarrow \mathcal{Y}_{n,k}$.

1.1. Main theorems. A *Bernardi polynomial* [1] is a map $B : \Gamma_{n,k} \rightarrow \mathbb{Q}[q, y, z]$ defined as

$$(1.1) \quad B_G(q, y, z) = \sum_{f: \{1, \dots, n\} \rightarrow \{1, \dots, q\}} y^{\#f_G^>} z^{\#f_G^<}$$

where $f_G^>$ (resp., $f_G^<$) is the set of edges $[ab]$ of G such that $f(b) > f(a)$ (resp., $f(b) < f(a)$). See [1] for a detailed analysis of the properties of B_G (in particular, for the proof of its polynomiality).

Bernardi polynomial is a directed version of the *full chromatic polynomial*, which is a map $C : \Upsilon_{n,k} \rightarrow \mathbb{Q}[q, y]$ defined as

$$C_G(q, y) = \sum_{f: \{1, \dots, n\} \rightarrow \{1, \dots, q\}} y^{\#f_G^\neq}$$

where f_G^\neq is the set of edges $[ab]$ of G such that $f(b) \neq f(a)$. The *classical Potts polynomial* (as defined e.g. in [7]) is related to the full chromatic polynomial by $Z_G(q, v) = (v+1)^k C_G(q, 1/(v+1))$; see [1] for details.

For any $G \in \Gamma_{n,k}$ (resp., $G \in \Upsilon_{n,k}$) we denote by \widehat{G} the graph G with all the loops deleted (and the numbering of the non-loop edges shifted accordingly); we have $\widehat{G} \in \Gamma_{n,k-\ell}$ (resp., $G \in \Upsilon_{n,k-\ell}$) where ℓ is the number of loops in G . It follows directly from the definition that $B_G(q, y, z) = B_{\widehat{G}}(q, y, z)$; in the undirected case $Z_G(q, v) = (v+1)^\ell Z_{\widehat{G}}(q, v)$.

The *universal Bernardi polynomial* is an element of $\mathcal{G}_{n,k}[q, y, z]$ defined as

$$\mathcal{B}_{n,k}(q, y, z) \stackrel{\text{def}}{=} \sum_{G \in \Gamma_{n,k}} B_G(q, y, z)G.$$

For a polynomial $P \in \mathbb{C}[q, y, z]$ denote by $[P]_k$ the sum of terms containing monomials $q^s y^i z^j$ with $i+j=k$ (and any s). The *universal truncated Bernardi polynomial* is an element of $\mathcal{G}_{n,k}[q, y, z]$ defined as

$$\widehat{\mathcal{B}}_{n,k}(q, y, z) \stackrel{\text{def}}{=} \sum_{G \in \Gamma_{n,k}} [B_G]_k(q, y, z)G.$$

Note that $[B_G]_k = 0$ if (and only if) G contains at least one loop; that is, $\widehat{\mathcal{B}}_{n,k}$ contains only loopless graphs. $\widehat{\mathcal{B}}_{n,k}$ is homogeneous of degree k with respect to y and z and is not homogeneous with respect to q .

The *universal Potts polynomial* and the *universal truncated Potts polynomial* are elements of $\mathcal{Y}_{n,k}[q, v]$ defined, respectively, as

$$\begin{aligned} \mathcal{Z}_{n,k}(q, v) &\stackrel{\text{def}}{=} \sum_{G \in \Gamma_{n,k}} Z_{\widehat{G}}(q, v)G, \\ \widehat{\mathcal{Z}}_{n,k}(q, v) &\stackrel{\text{def}}{=} \sum_{G \in \Gamma_{n,k} \text{ has no loops}} Z_G(q, v)G. \end{aligned}$$

For any $i = 1, \dots, k$ and $p, q = 1, \dots, n$ denote by $R_{p,q;i} : \Gamma_{n,k} \rightarrow \Gamma_{n,k}$ the map replacing the edge number i of every graph $G \in \Gamma_{n,k}$ by the edge $[p, q]$ carrying the same number i . Also denote by $B_i : \mathcal{G}_{n,k} \rightarrow \mathcal{G}_{n,k}$ the linear operator acting on the graph $G \in \Gamma_{n,k}$ such that $[a, b]$ is its edge number i as

$$B_i(G) = \begin{cases} G, & a \neq b, \\ -\sum_{m \neq a} R_{a,m;i}G, & a = b. \end{cases}$$

Following [2] call the product $\Delta \stackrel{\text{def}}{=} B_1 \dots B_k : \mathcal{G}_{n,k} \rightarrow \mathcal{G}_{n,k}$ the *Laplace operator*. The undirected version of the Laplace operator is defined as follows: if $G \in \Upsilon_{n,k}$ then $\Delta(G) \stackrel{\text{def}}{=} |\Delta(\Phi)|$ where $\Phi \in \Gamma_{n,k}$ is any directed graph such that $G = |\Phi|$.

The main results of this paper are the following two theorems:

Theorem 1.1.

$$\Delta \mathcal{B}_{n,k}(q, y, z) = \widehat{\mathcal{B}}_{n,k}(q, y-1, z-1).$$

and its undirected version

Theorem 1.2.

$$\Delta \mathcal{Z}_{n,k}(q, v) = (-1)^k \widehat{\mathcal{Z}}_{n,k}(q, -v).$$

1.2. Corollaries.

1.2.1. *Universal chromatic polynomials.* Following [1], denote by χ_G^{\geq} (a *chromatic polynomial* of the directed graph $G \in \Gamma_{n,k}$) a polynomial such that for any $q = 1, 2, \dots$ the value $\chi_G^{\geq}(q)$ is equal to the number of mappings $f : \{1, \dots, n\} \rightarrow \{1, \dots, q\}$ such that $f(a) \geq f(b)$ for every edge $[ab] \in G$. Also denote by $\chi_G^>$ (a *strict chromatic polynomial* of G) a polynomial such that for any $q = 1, 2, \dots$ the value $\chi_G^>(q)$ is equal to the number of mappings $f : \{1, \dots, n\} \rightarrow \{1, \dots, q\}$ such that $f(a) > f(b)$ for every edge $[ab] \in G$.

Comparing these definitions with the definition of the Bernardi polynomial in Section 1.1 one obtains the equalities:

$$\begin{aligned}\chi_G^{\geq}(q) &= B_G(q, 0, 1), \\ \chi_G^>(q) &= [B_G]_k(q, 0, 1).\end{aligned}$$

Thus, one may call the elements of $\mathcal{G}_{n,k}$

$$\mathcal{X}_{n,k}^{\geq}(q) \stackrel{\text{def}}{=} \mathcal{B}_{n,k}(q, 0, 1) = \sum_{G \in \Gamma_{n,k}} \chi_G^{\geq}(q) G,$$

and

$$\mathcal{X}_{n,k}^>(q) \stackrel{\text{def}}{=} \widehat{\mathcal{B}}_{n,k}(q, 0, 1) = \sum_{G \in \Gamma_{n,k}} \chi_G^>(q) G$$

universal chromatic polynomials. Substitution of $y = 0$ and $z = 1$ in Theorem 1.1 yields

Corollary 1.3. $\Delta \mathcal{X}_{n,k}^{\geq}(q) = (-1)^k \mathcal{X}_{n,k}^>(q).$

1.2.2. *Higher matrix-tree theorems.* A graph $G \in \Gamma_{n,k}$ is called *acyclic* if it contains no oriented cycles (in particular, no loops); G is called *totally cyclic* (or *strongly semiconnected*, following the terminology of [2]) if every edge of G is a part of a directed cycle.

It is possible to make further specialization of parameters in Corollary 1.3 due to the following

Proposition 1.4. *For any $G \in \Gamma_{n,k}$*

$$(1.2) \quad \chi_G^{\geq}(-1) = \begin{cases} (-1)^{\beta_0(G)}, & \text{if } G \text{ is totally cyclic,} \\ 0 & \text{otherwise,} \end{cases}$$

$$(1.3) \quad \chi_G^>(-1) = \begin{cases} (-1)^k, & \text{if } G \text{ is acyclic,} \\ 0 & \text{otherwise.} \end{cases}$$

where $\beta_0(G)$ is the 0-th Betti number (i.e. the number of connected components) of the graph G .

For proof see [1, Eq. (45) and Definition 5.1]. Note that it follows immediately from the definition that $\chi_G^{\geq} \equiv 0$ if (and only if) G contains an oriented cycle (e.g. a loop), and that $\chi_G^{\geq}(q) = q^{\beta_0(G)}$ if G is totally cyclic, so one half of each formula is evident (but not the other half).

Consider now (following [2]) the sum

$$\det_{n,k} \stackrel{\text{def}}{=} \frac{(-1)^k}{k!} \chi_G^{\geq}(-1) = \frac{(-1)^k}{k!} \sum_{G \in \mathfrak{S}_{n,k}} (-1)^{\beta_0(G)} G$$

where by $\mathfrak{S}_{n,k} \subset \Gamma_{n,k}$ we denote the set of all totally cyclic graphs. Thus, Corollary 1.3 specializes to

Corollary 1.5. $\Delta \det_{n,k} = \frac{(-1)^n}{k!} \sum_{G \in \mathfrak{A}_{n,k}} G.$

where $\mathfrak{A}_{n,k} \subset \Gamma_{n,k}$ is the set of all acyclic graphs.

Corollary 1.5 admits a refinement. A totally cyclic graph may have isolated vertices (the ones not incident to any edge). Let $I = \{i_1 < \dots < i_s\} \subset \{1, \dots, n\}$ be a set of vertices. We call a *diagonal I-minor* the element

$$(1.4) \quad \det_{n,k}^I \stackrel{\text{def}}{=} \frac{(-1)^k}{k!} \sum_{G \in \mathfrak{S}_{n,k}^I} (-1)^{\beta_0(G)} G \in \mathcal{G}_{n,k}$$

where $\mathfrak{S}_{n,k}^I$ is the set of all totally cyclic graphs $G \in \Gamma_{n,k}$ such that the vertices i_1, \dots, i_s , and only they, are isolated. Similarly, denote by $\mathfrak{A}_{n,k}^I \subset \Gamma_{n,k}$ the set of all acyclic graphs such that i_1, \dots, i_s , and only they, are sinks (vertices without edges starting at them); so Corollary 1.5 now looks like

$$\Delta \sum_{I \subset \{1, \dots, n\}} \det_{n,k}^I = \frac{(-1)^n}{k!} \sum_{I \subset \{1, \dots, n\}} \sum_{G \in \mathfrak{A}_{n,k}^I} G.$$

It follows from the definition of the Laplace operator that if $G \in \mathfrak{S}_{n,k}^I$ then $\Delta G = \sum_H x_H H$ where $x_H \in \mathbb{Z}$ and all H have i_1, \dots, i_s , and only them, as sinks. Since $\mathfrak{S}_{n,k}^I$ with different I do not intersect, and the same is true for $\mathfrak{A}_{n,k}^I$, there is

Corollary 1.6 (of Corollary 1.5). *For every $I = \{i_1, \dots, i_s\} \subset \{1, \dots, n\}$ one has $\Delta \det_{n,k}^I = \frac{(-1)^n}{k!} \sum_{G \in \mathfrak{A}_{n,k}^I} G.$*

This is the abstract-matrix tree theorem [2, Theorem 1.7] which, in turn, is a higher-degree generalization of the celebrated matrix-tree theorem (first discovered by G. Kirchhoff in 1847 [3] and extended to the directed graphs by W. Tutte [6]).

Acknowledgements. The research was funded by the Russian Academic Excellence Project ‘5-100’ and by the grant No. 15-01-0031 “Hurwitz numbers and graph isomorphism” of the Scientific Fund of the Higher School of Economics.

2. PROOFS

A graph $H \in \Gamma_{n,m}$ is called a subgraph of $G \in \Gamma_{n,k}$ (notation $H \subseteq G$) if it can be obtained from G by deletion of several edges. (When one deletes the edge number s from the graph, the numbers of the remaining edges are preserved if they are less than s and are lowered by 1 if they are greater than s .)

For convenience denote by $e(G)$ the number of edges of the graph G (so $e(G) = k$ if $G \in \Gamma_{n,k}$). Proof of Theorem 1.1 involves the following well-known lemma:

Lemma 2.1 (Moebius inversion formula, [4]). *Let $f : \bigcup_k \Gamma_{n,k} \rightarrow \mathbb{C}$ be a function on the set of graphs with n vertices, and let the function h on the same set be defined by the equality $h(G) = \sum_{H \subseteq G} f(H)$ for every $G \in \Gamma_{n,k}$. Then one has $(-1)^{e(G)} f(G) = \sum_{H \subseteq G} (-1)^{e(H)} h(H).$*

Proof of Theorem 1.1. Let $\Delta \mathcal{B}_{n,k}(q, y, z) = \sum_{G \in \Gamma_{n,k}} x_G G$; by the definition of the Laplace operator $x_G \neq 0$ only if G contains no loops. For a graph $H \in \Gamma_{n,k}$ the element $\Delta H \in \mathcal{G}_{n,k}$ contains a term $y_{G,H} G$ with $y_{G,H} \neq 0$ if and only if $\Phi \stackrel{\text{def}}{=} \widehat{H}$

(the graph H with all the loops deleted) is a subgraph of G . For any subgraph $\Phi \subseteq G$ of a loopless graph G there exists exactly one $H \stackrel{\text{def}}{=} L(\Phi)$ such that $\Phi = \widehat{H}$: every edge $[ab]$ present in G but missing in Φ is replaced by the loop $[aa]$ in H .

Eventually, the coefficient $y_{G,H}$ in this case is

$$y_{G,H} = (-1)^{\#\text{of loops in } H} B_H(q, y, z) = (-1)^{k-e(\Phi)} B_\Phi(q, y, z).$$

where $\Phi = \widehat{H}$.

By [1, Eq. (21)] one has $\sum_{\Phi \subseteq G} [B_\Phi]_{e(\Phi)}(q, y-1, z-1) = B_G(q, y, z)$. Applying the Moebius inversion formula (Proposition 2.1) to this identity one obtains

$$x_G = \sum_{\Phi \subseteq G} y_{G,L(\Phi)} = (-1)^k \sum_{\Phi \subseteq G} (-1)^{e(\Phi)} B_\Phi(q, y, z) = [B_G]_k(q, y-1, z-1).$$

□

Proof of Theorem 1.2. is similar to that of Theorem 1.1: again, if $\Delta \mathcal{Z}_{n,k}(q, v) = \sum_G x_G G$ then G entering the sum have no loops. A contribution $y_{G,H}$ of a graph H that into x_G is nonzero if and only if $\Phi = \widehat{H}$ is a subgraph of G . For a subgraph $\Phi \subseteq G$ having $e(\Phi)$ edges there are $2^{k-e(\Phi)}$ graphs H such that $\Phi = \widehat{H}$: every edge $[ab]$ present in G but missing in Φ may correspond either to a loop $[aa]$ or to a loop $[bb]$ in H ; recall that $a \neq b$ because G is loopless.

The contribution $y_{G,H}$ of all such graphs H into x_G is the same and is equal to $(-1)^{k-e(\Phi)} Z_\Phi(q, v)$. Now by [5] one has $Z_G(q, v) = \sum_{H \subseteq G} q^{\beta_0(H)} v^{e(H)}$, and therefore

$$\begin{aligned} x_G &= \sum_{\Phi \subseteq G} 2^{k-e(\Phi)} (-1)^{k-e(\Phi)} Z_\Phi(q, v) = (-2)^k \sum_{\Phi \subseteq G} \left(-\frac{1}{2}\right)^{e(\Phi)} Z_\Phi(q, v) \\ &= (-2)^k \sum_{\Psi \subseteq \Phi \subseteq G} \left(-\frac{1}{2}\right)^{e(\Phi)} q^{\beta_0(\Psi)} v^{e(\Psi)} = (-2)^k \sum_{\Psi \subseteq G} q^{\beta_0(\Psi)} v^{e(\Psi)} \sum_{\Phi \supseteq \Psi} \left(-\frac{1}{2}\right)^{e(\Phi)} \\ &= (-2)^k \sum_{\Psi \subseteq G} q^{\beta_0(\Psi)} v^{e(\Psi)} \left(-\frac{1}{2}\right)^{e(\Psi)} \left(1 - \frac{1}{2}\right)^{k-e(\Psi)} = (-1)^k Z_G(q, -v). \end{aligned}$$

□

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